

# Some Exact Solutions to Equations of Motion of an Incompressible Third Grade Fluid

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This investigation deals with some exact solutions of the equations governing the steady plane motions of an incompressible third grade fluid by using complex variables and complex functions. Some of the solutions admit, as particular cases, all the solutions of Moro et al.

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## I. INTRODUCTION

The governing equations that describe flows of Newtonian fluids are the Navier-Stokes equations. The mechanical behavior of many real fluids, especially those of low molecular weight, appears to be accurately described by these equations over a wide range of circumstances. There are, however, many real substances which are capable of flowing but which are not at all well described by the Navier-Stokes theory. Due to this reason many fluid models have been proposed and studied by different authors. Among these, the fluids of differential type have received much attention. Interesting studies of differential type fluids are given by Rajagopal [1], Erdogan [2], Rajagopal and Gupta [3], Bandelli [4], Siddiqui and Kaloni [5], Benharbit and Siddiqui [6], Ariel [7, 8], Fetecau and Fetecau [9] and Hayat et al. [10, 11, 12]. The fluids of third grade, which form a subclass of the fluids of differential type, have been successfully studied in various types of motions.

Moro et al. [13] determined some exact solutions of the equations governing the steady plane motion of an incompressible third grade fluid employing hodograph and Legendre transformations.

The aim of this paper is to present some exact solutions for steady plane motions of incompressible third grade fluids. For this, the complex variables and complex functions are used. Some previous solutions can be obtained as special cases of our solutions. In sections 2 and 3 are presented the constitutive equations and the expression of the vorticity function  $\omega$ . Section 4 contains solutions and their graphical illustrations.

## II. CONSTITUTIVE AND GOVERNING EQUATIONS

The fluids of grade  $n$ , introduced by Rivlin and Ericksen [14], are the fluids for which the stress tensor is

a polynomial of degree  $n$  in the first  $n$  Rivlin-Ericksen tensors defined recursively by

$$A_1 = (\partial_i u_j + \partial_j u_i)_{i,j};$$

$$A_n = \frac{d}{dt} A_{n-1} + A_{n-1} L + L^t A_{n-1}, \quad n > 1 \quad (1)$$

where  $\frac{d}{dt} = \partial_t + u \cdot \nabla$  denotes the material derivative and

$$L = (\partial_j u_i)_{i,j}; \quad L^t = (\partial_i u_j)_{i,j}.$$

Physical considerations were taken into account by Fosdick and Rajagopal [15] in order to obtain the following form of constitutive equation for the Cauchy stress  $T$  in a third grade fluid:

$$T = -pI + \mu A_1 + \alpha_1 A_2 + \alpha_2 A_1^2 + \beta_1 A_3 + \beta_2 [A_1 A_2 + A_2 A_1] + \beta_3 (tr A_1^2) A_1 \quad (2)$$

where  $-pI$  is the spherical stress due to the constraint of incompressibility,  $\mu$  is the dynamic viscosity,  $\alpha_i (i = 1, 2)$  and  $\beta_i (i = 1, 2, 3)$  are the material constants and the three Rivlin-Ericksen tensors  $A_1, A_2$  and  $A_3$  are given in equation (1).

Furthermore, a complete thermodynamic analysis of the constitutive equation (2) has been given by Fosdick and Rajagopal [15]. The Clausius–Duhem inequality and the assumption that the Helmholtz free energy is a minimum in equilibrium provide the following restrictions,

$$\mu \geq 0, \quad \alpha_1 \geq 0, \quad |\alpha_1 + \alpha_2| \leq \sqrt{24\mu\beta_3},$$

$$\beta_1 = \beta_2 = 0, \quad \beta_3 \geq 0. \quad (3)$$

Thus equation (2) becomes

$$T = -pI + \alpha_1 A_2 + \alpha_2 A_1^2 + [\mu + \beta_3 (tr A_1^2)] A_1,$$

where  $\mu_{eff} = \mu + \beta_3 (tr A_1^2)$  is the effective shear-dependent viscosity.

The velocity field corresponding to the motion is given as:

$$v = v(x, y) = (u(x, y), v(x, y), 0).$$

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### III. FLOW EQUATIONS

The basic equations governing the steady plane motion of a homogeneous incompressible fluid of third grade, in the absence of body forces are [13]

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (4)$$

$$\begin{aligned} \frac{\partial h}{\partial x} = & \rho v \omega - \mu \frac{\partial \omega}{\partial y} - \alpha_1 v \nabla^2 \omega - \beta_3 \frac{\partial(\omega M)}{\partial y} + \\ & + 2\beta_3 \left( \frac{\partial u}{\partial x} \frac{\partial M}{\partial x} + \frac{\partial v}{\partial x} \frac{\partial M}{\partial y} \right) \end{aligned} \quad (5)$$

$$\begin{aligned} \frac{\partial h}{\partial y} = & -\rho u \omega + \mu \frac{\partial \omega}{\partial x} + \alpha_1 u \nabla^2 \omega + \beta_3 \frac{\partial(\omega M)}{\partial x} + \\ & + 2\beta_3 \left( \frac{\partial u}{\partial y} \frac{\partial M}{\partial x} + \frac{\partial v}{\partial y} \frac{\partial M}{\partial y} \right) \end{aligned} \quad (6)$$

where

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

$$h = \frac{\rho q^2}{2} - \alpha_1 (u \nabla^2 u + v \nabla^2 v) - (3\alpha_1 + 2\alpha_2) \frac{M}{4} + p \quad (7)$$

$$M = 4\left(\frac{\partial u}{\partial x}\right)^2 + 4\left(\frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2$$

$$q^2 = u^2 + v^2$$

Equations (5) and (6) are non-linear partial differential equations for three unknowns  $u$ ,  $v$  and  $\rho$  as functions of  $x$  and  $y$ . In equations (5), (6) and (7), the viscosity  $\mu$  and the material constants  $\alpha_1$ ,  $\alpha_2$ ,  $\beta_1$ ,  $\beta_2$ ,  $\beta_3$  satisfy the constraints given in equation (3) [15, 16].

Equation (4) implies the existence of a stream function  $\psi(x, y)$  such that

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x} \quad (8)$$

Equations (4) and (5), on utilizing equation (8) and the compatibility condition  $\frac{\partial^2 h}{\partial x \partial y} = \frac{\partial^2 h}{\partial y \partial x}$ , yield

$$\begin{aligned} & \rho \left( \frac{\partial \psi}{\partial y} \frac{\partial \omega}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial \omega}{\partial y} \right) - \alpha_1 \left\{ \frac{\partial \psi}{\partial y} \frac{\partial(\nabla^2 \omega)}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial(\nabla^2 \omega)}{\partial y} \right\} \\ & - \beta_3 \left\{ \frac{\partial^2(\omega M)}{\partial x^2} + \frac{\partial^2(\omega M)}{\partial y^2} \right\} + 2\beta_3 \left\{ 2 \frac{\partial^2 \psi}{\partial x \partial y} \frac{\partial^2 M}{\partial y \partial x} - \right. \\ & \left. - \frac{\partial^2 \psi}{\partial x^2} \frac{\partial^2 M}{\partial y^2} - \frac{\partial^2 \psi}{\partial y^2} \frac{\partial^2 M}{\partial x^2} \right\} - \mu \nabla^2 \omega = 0 \end{aligned} \quad (9)$$

where

$$\omega = -\left( \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right)$$

$$M = 8 \frac{\partial^2 \psi^2}{\partial y \partial x} + 2 \left( \frac{\partial^2 \psi}{\partial y^2} - \frac{\partial^2 \psi}{\partial x^2} \right)$$

Let

$$z = x + iy \quad \text{and} \quad \bar{z} = x - iy$$

Then from Stallybrass [17]

$$\begin{aligned} 2 \frac{\partial(\bullet)}{\partial \bar{z}} &= \frac{\partial(\bullet)}{\partial x} + i \frac{\partial(\bullet)}{\partial y}, \quad 2 \frac{\partial(\bullet)}{\partial z} = \frac{\partial(\bullet)}{\partial x} - i \frac{\partial(\bullet)}{\partial y}, \\ 4Im \left\{ \frac{\partial(\bullet)}{\partial \bar{z}} \frac{\partial(\circ)}{\partial z} \right\} &= \frac{\partial(\bullet)}{\partial y} \frac{\partial(\circ)}{\partial x} - \frac{\partial(\bullet)}{\partial x} \frac{\partial(\circ)}{\partial y} \end{aligned} \quad (10)$$

Equation (9), on utilizing equation (10), becomes

$$\begin{aligned} & Im \left\{ \frac{\partial \psi}{\partial \bar{z}} \left( \rho \frac{\partial \omega}{\partial z} - \alpha_1 \frac{\partial^3 \omega}{\partial z^2 \partial \bar{z}} \right) \right\} - \mu \frac{\partial^2 \omega}{\partial z \partial \bar{z}} - \\ & - 2\beta_3 \left\{ \frac{\partial^2 \psi}{\partial \bar{z}^2} \frac{\partial^2 M}{\partial \bar{z}^2} + \frac{\partial^2 \psi}{\partial z^2} \frac{\partial^2 M}{\partial z^2} \right\} - \\ & - \beta_3 \left\{ \frac{\partial^2 \omega}{\partial z \partial \bar{z}} M + \frac{\partial \omega}{\partial z} \frac{\partial M}{\partial \bar{z}} + \frac{\partial \omega}{\partial \bar{z}} \frac{\partial M}{\partial z} \right\} = 0 \end{aligned} \quad (11)$$

where

$$\omega = -4 \frac{\partial^2 \psi}{\partial z \partial \bar{z}} \quad (12)$$

$$M = 32 \frac{\partial^2 \psi}{\partial \bar{z}^2} \frac{\partial^2 \psi}{\partial z^2}$$

### IV. SOLUTIONS

On integrating equation (12), we obtain the general form of the exact solutions

$$\psi = -\frac{1}{4} \int \int \omega \, dz \, d\bar{z} + A + \bar{A} \quad (13)$$

where  $A$  and  $\bar{A}$  are complex functions and  $\bar{A}$  is the complex conjugate of  $A$ .

To determine the solution of equation (11), our strategy will be to specify the vorticity function  $\omega$  and determine the condition, which the functions  $A$  and  $\bar{A}$  must satisfy. The condition is obtained from equation (11) utilizing equation (13).

I. When vorticity  $\omega$  is constant say  $\omega_0$ , then equation (13) yields

$$\psi = -\frac{1}{4} \omega_0 z \bar{z} + A + \bar{A} \quad (14)$$

Equation (11), on utilizing equation (14), yields

$$\beta_3 \left( \frac{\partial^4 A}{\partial z^4} + \frac{\partial^4 \bar{A}}{\partial \bar{z}^4} \right) = 0, \quad \frac{\partial^2 A}{\partial z^2} \neq 0, \quad \frac{\partial^2 \bar{A}}{\partial \bar{z}^2} \neq 0.$$

When  $\beta_3 \neq 0$ , then

$$A = \frac{\iota a_1 z^4}{24} + \frac{a_2 z^3}{6} + \frac{a_3 z^2}{2} + a_4 z + a_5 \quad (15)$$

where  $a_i$ 's are all complex constants. The velocity components  $u$  and  $v$  are given by

$$u = \frac{\iota \omega_0}{4}(z - \bar{z}) - \iota \left\{ -\frac{\iota \bar{a}_1 z^3}{6} + \frac{\bar{a}_2 z^2}{3} + \bar{a}_3 z + \bar{a}_4 - \frac{\iota a_1 z^3}{6} - \frac{a_2 z^2}{3} - a_3 z - a_4 \right\};$$

$$v = \frac{\omega_0}{4}(z + \bar{z}) - \left\{ \frac{\iota a_1 z^3}{6} + \frac{a_2 z^2}{3} + a_3 z + a_4 - \frac{\iota \bar{a}_1 z^3}{6} + \frac{\bar{a}_2 z^2}{3} + \bar{a}_3 z + \bar{a}_4 \right\}$$

Equation (14), on utilizing equation (15), yields

$$\psi = -\frac{\omega_0 z \bar{z}}{4} + \frac{\iota(a_1 z^4 - \bar{a}_1 \bar{z}^4)}{24} + \frac{(a_2 z^3 + \bar{a}_2 \bar{z}^3)}{6} + \frac{(a_3 z^2 + \bar{a}_3 \bar{z}^2)}{2} + (a_4 z + \bar{a}_4 \bar{z}) + a \quad (16)$$

where  $a = a_5 + \bar{a}_5$ .

The stream function  $\psi$  in equation (16) is represented graphically in Fig.1 with  $\omega_0 = -1$ ,  $a_1 = 1 + 2\iota$ ,  $a_2 = 1 + \iota$ ,  $a_3 = 1 + 5\iota$ ,  $a_4 = 2 + 0.5\iota$ ,  $a = 2$  and  $x, y \in [-1, 1]$ .

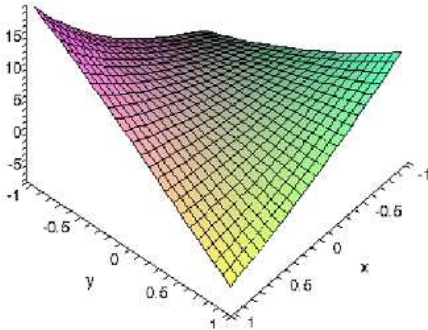


Figure No. 1: Graphical representation of equation (16)

When  $\beta_3 = 0$ , then equation (15) is identically satisfied and the complex  $A$  becomes arbitrary, due to which  $A$  enables us to construct a large number of streamfunction  $\psi$  and hence a large number of solutions to the flow equations. We mention that by taking  $\omega = 0$  or appropriately choosing complex constants  $a_3$ ,  $a_4$ ,  $a_5$ , and  $a_1 = 0 = a_2$  in equation (15), or by taking  $A = \iota a \log z$  (for  $\beta_3 = 0$ ), we get all the solutions of Moro et al. [13].

Further more, if we take the function  $A = (c_1 + \iota c_2)z^2$  or  $A = (c_1 + \iota c_2)\ln z$  and choose appropriately the constants or apply the appropriate boundary conditions we get the plane Couette flow, the flow due to a spiral vortex at the origin and the flows having streamlines as a family of ellipses, concentric circles, rectangular hyperbolae.

II. When  $\omega$  is non-constant, the solutions of equation (11) are determined as follows:

(i) When  $\omega = m_1 z + \bar{m}_1 \bar{z}$ , the equation (11) yields

$$\begin{aligned} \text{Im} \left\{ -\frac{m_1^2 z^2}{8} + m_1 \frac{\partial \bar{A}}{\partial \bar{z}} \right\} = \lambda \left\{ 6m_1 \bar{m}_1 (m_1 z + \bar{m}_1 \bar{z}) - \right. \\ \left. -8 \left( m_1^2 \frac{\partial^2 \bar{A}}{\partial \bar{z}^2} + \bar{m}_1^2 \frac{\partial^2 A}{\partial z^2} \right) - 16 \left( m_1^2 \bar{z} \frac{\partial^3 \bar{A}}{\partial \bar{z}^3} + \bar{m}_1^2 z \frac{\partial^3 A}{\partial z^3} \right) \right. \\ \left. + 64 \left( m_1 \frac{\partial^3 \bar{A}}{\partial \bar{z}^3} \frac{\partial^2 A}{\partial z^2} + \bar{m}_1 \frac{\partial^3 A}{\partial z^3} \frac{\partial^2 \bar{A}}{\partial \bar{z}^2} \right) + \right. \\ \left. + 64 \frac{\partial^3 A}{\partial z^3} \frac{\partial^3 \bar{A}}{\partial \bar{z}^3} (m_1 z + \bar{m}_1 \bar{z}) \right\} \end{aligned} \quad (17)$$

where  $\lambda = \frac{\beta_3}{\rho}$ , and  $m_1$  is the complex constant. The L.H.S of equation (17) suggests to assume

$$\frac{\partial \bar{A}}{\partial \bar{z}} = \lambda_1 \bar{z}^2 + \lambda_2 \bar{z} + \lambda_3$$

This on putting in equation (17), gives

$$\lambda_1 = -\frac{\bar{m}_1^2}{8m_1}, \quad \lambda_2 = 40\iota \lambda \bar{m}_1^2, \quad \lambda_3 = 0.$$

The solution of equation (17), therefore, is

$$A = -\frac{m_1^2 z^3}{24\bar{m}_1} - 20\iota \lambda m_1^2 z^2 + m_2 \quad (18)$$

where  $m_2$  is a complex constant.

Equation (13), using (18) becomes

$$\begin{aligned} \psi = -\frac{z \bar{z}}{8} (m_1 z + \bar{m}_1 \bar{z}) - \frac{(m_1^3 z^3 + \bar{m}_1^3 \bar{z}^3)}{24m_1 \bar{m}_1} + \\ + 20\iota \lambda (\bar{m}_1^2 z^2 - m_1^2 \bar{z}^2) + m \end{aligned} \quad (19)$$

where  $m = m_2 + \bar{m}_2$ .

The stream function  $\psi$  in equation (19) is represented graphically in Fig.2 with  $m_1 = 1 + 2\iota$ ,  $m = 1$ ,  $\lambda = 0.3$  and  $x, y \in [-1, 1]$ .

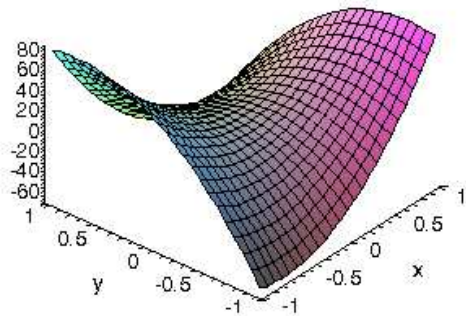


Figure No. 2: Graphical representation of equation (19).

(ii) For  $\omega = B(z + \bar{z})$ , the constant  $B$  being real, the equation (11) becomes

$$\begin{aligned} \text{Im}\left\{-\frac{Bz^2}{8} + \frac{\partial \bar{A}}{\partial \bar{z}}\right\} = \lambda \left\{ 6B^2(z + \bar{z}) - \right. \\ -8B\left(\frac{\partial^2 A}{\partial z^2} + \frac{\partial^2 \bar{A}}{\partial \bar{z}^2}\right) - 16B\left(\bar{z}\frac{\partial^3 \bar{A}}{\partial \bar{z}^3} + z\frac{\partial^3 A}{\partial z^3}\right) + \\ + 64\left(\frac{\partial^2 A}{\partial z^2}\frac{\partial^3 \bar{A}}{\partial \bar{z}^3} + \frac{\partial^2 \bar{A}}{\partial \bar{z}^2}\frac{\partial^3 A}{\partial z^3}\right) + \\ + 64\frac{\partial^3 A}{\partial z^3}\frac{\partial^3 \bar{A}}{\partial \bar{z}^3}(z + \bar{z}) - 4B\left(z^2\frac{\partial^4 A}{\partial z^4} + \bar{z}^2\frac{\partial^4 \bar{A}}{\partial \bar{z}^4}\right) + \\ + 32\left(z\frac{\partial^2 \bar{A}}{\partial \bar{z}^2}\frac{\partial^4 A}{\partial z^4} + \bar{z}\frac{\partial^2 A}{\partial z^2}\frac{\partial^4 \bar{A}}{\partial \bar{z}^4}\right) - \\ \left. - \frac{64}{B}\left(\frac{\partial^2 A^2}{\partial z^2}\frac{\partial^4 \bar{A}}{\partial \bar{z}^4} + \frac{\partial^2 \bar{A}^2}{\partial \bar{z}^2}\frac{\partial^4 A}{\partial z^4}\right)\right\} \end{aligned} \quad (20)$$

The L.H.S of equation (20) suggests  $\frac{\partial \bar{A}}{\partial \bar{z}}$  to be a polynomial in  $\bar{z}$  of degree two and therefore on substituting

$$\frac{\partial \bar{A}}{\partial \bar{z}} = l_1 \bar{z}^2 + l_2 \bar{z} + \lambda_4,$$

in equation (20), we get

$$l_1 = -\frac{B}{8}, \quad l_2 = 40\iota\lambda B^2, \quad \lambda_4 = 0$$

Hence

$$\bar{A} = -\frac{Bz^3}{24} + 20\iota\lambda B^2 \bar{z} + m_3 \quad (21)$$

where  $m_3$  is an arbitrary complex constant. Using equation (21), equation (13) implies,

$$\psi = -\frac{Bz\bar{z}}{8}(z + \bar{z}) - \frac{B}{24}(z^3 + \bar{z}^3) + 20\iota B^2 \lambda (\bar{z} - z) + n \quad (22)$$

where  $n = m_3 + \bar{m}_3$ .

The stream function  $\psi$  in equation (22) is represented graphically in Fig.3 with  $B = -2$ ,  $\lambda = 2$ ,  $n = 1$  and  $x \in [-10, 10]$ ,  $y \in [0, 10]$ .

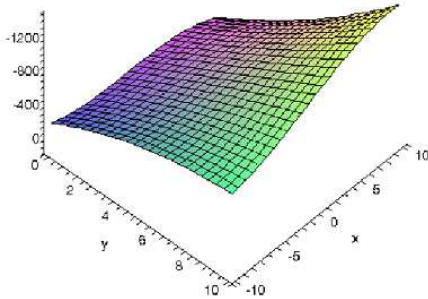


Figure No. 3: Graphical representation of equation (22).

(iii) When  $\omega = D(z + \bar{z} + E)$ ,  $D$  and  $E$  being real,

then equation (11) yields

$$\begin{aligned} \text{Im}\left\{-\frac{Dz^2}{8} - \frac{DEz}{4} + \frac{\partial \bar{A}}{\partial \bar{z}}\right\} = \lambda \left\{ 6D^2(z + \bar{z}) - \right. \\ -8D\left(\frac{\partial^2 A}{\partial z^2} + \frac{\partial^2 \bar{A}}{\partial \bar{z}^2}\right) - 16D\left(z\frac{\partial^3 A}{\partial z^3} + \bar{z}\frac{\partial^3 \bar{A}}{\partial \bar{z}^3}\right) + \\ + 64\left(\frac{\partial^2 A}{\partial z^2}\frac{\partial^3 \bar{A}}{\partial \bar{z}^3} + \frac{\partial^2 \bar{A}}{\partial \bar{z}^2}\frac{\partial^3 A}{\partial z^3}\right) + 64\left(z\frac{\partial^3 A}{\partial z^3}\frac{\partial^3 \bar{A}}{\partial \bar{z}^3} + \right. \\ \left. + \bar{z}\frac{\partial^3 \bar{A}}{\partial \bar{z}^3}\frac{\partial^3 A}{\partial z^3}\right) + 4D^2E + 64E\frac{\partial^3 A}{\partial z^3}\frac{\partial^3 \bar{A}}{\partial \bar{z}^3} - \\ - 4D\left(\bar{z}^2\frac{\partial^4 \bar{A}}{\partial \bar{z}^4} + z^2\frac{\partial^4 A}{\partial z^4}\right) - \frac{64}{D}\left(\frac{\partial^2 A^2}{\partial z^2}\frac{\partial^4 \bar{A}}{\partial \bar{z}^4} + \right. \\ \left. + \frac{\partial^2 \bar{A}^2}{\partial \bar{z}^2}\frac{\partial^4 A}{\partial z^4}\right) + 32\left(\frac{\partial^2 \bar{A}^2}{\partial \bar{z}^2}\frac{\partial^4 A}{\partial z^4} + \frac{\partial^2 A^2}{\partial z^2}\frac{\partial^4 \bar{A}}{\partial \bar{z}^4}\right)\left\} \end{aligned}$$

Following the same procedure as that of previous case we find

$$\bar{A} = -\frac{Dz^3}{24} - \frac{DEz^2}{8} + 20\iota\lambda D^2 \bar{z}^2 + 20\iota\lambda D^2 E \bar{z} + m_4 \quad (23)$$

where  $m_4$  is an arbitrary complex constant.

Using (23), equation (13) becomes

$$\begin{aligned} \psi = -\frac{Dz\bar{z}}{8}(z + \bar{z} + 2E) - \frac{D}{24}(z^3 + \bar{z}^3) - \frac{DE}{8}(z^2 + \\ + \bar{z}^2) + 20\iota\lambda D^2(\bar{z}^2 - z^2) + 20\iota\lambda D^2 E(\bar{z} - z) + q \quad (24) \end{aligned}$$

where  $q = m_4 + \bar{m}_4$ .

The stream function  $\psi$  in equation (24) is represented graphically in Fig.4 with  $D = E = 1$ ,  $\lambda = 2$ ,  $q = -1$  and  $x, y \in [-10, 10]$ .

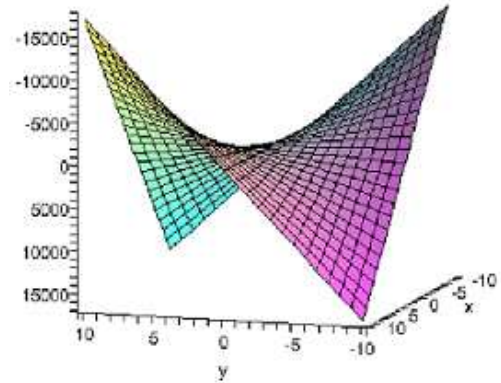


Figure No. 4: Graphical representation of equation (24).

(iv) When  $\omega = B\iota(z - \bar{z})$ , then equation (11) is satisfied provided

$$\bar{A} = -\frac{B\iota z^3}{24} - 20\iota\lambda B^2 \bar{z}^3 + m_5$$

where  $m_5$  is an arbitrary complex constant and the streamfunction  $\psi$  is given by

$$\psi = -\frac{B\iota}{8}z\bar{z}(z - \bar{z}) + \frac{B\iota}{24}(z^3 - \bar{z}^3) + 20\iota\lambda B^2(z^3 - \bar{z}^3) + r \quad (25)$$

where  $r = m_5 + \overline{m_5}$ .

The stream function  $\psi$  in equation (25) is represented graphically in Fig.5 with  $B = -5$ ,  $\lambda = 3$ ,  $r = 10$  and  $x \in [-2, 10]$ ,  $y \in [0, 8]$ .

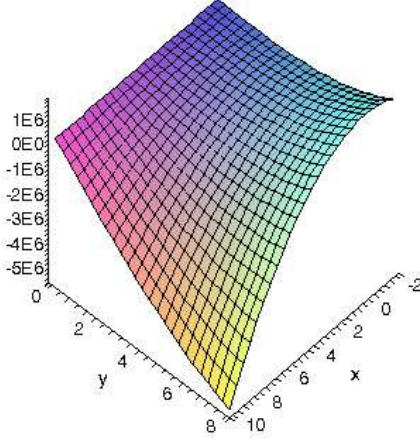


Figure No. 5: Graphical representation of equation (25).

( v ) For  $\omega = B \ln(z\bar{z}) + D_1$ , the equation (11) is satisfied provided

$$\overline{A} = m_6 \ln \bar{z} + m_7 \quad (26)$$

where  $m_6$ ,  $B$ ,  $D_1$ , are real constants and  $m_7$  is complex arbitrary constant.

According to equation (26), the streamfunction  $\psi$  becomes

$$\psi = -\frac{B}{4} z \bar{z} \left\{ \ln(z \bar{z}) - 2 \right\} - \frac{D_1}{4} z \bar{z} + m_6 \ln(z \bar{z}) + s \quad (27)$$

where  $s = m_7 + \overline{m_7}$ .

The stream function  $\psi$  in equation (27) is represented graphically in Fig.6 with  $m_6 = 2$ ,  $s = 4$  and  $x \in [-10, 2]$ ,  $y \in [-1, 2]$ .

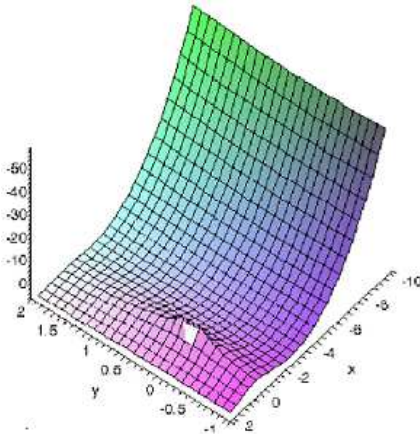


Figure No. 6: Graphical representation of equation (27).

The stream function  $\psi$  in equation (27) represents the motion whose streamlines are concentric circles with

constant speed along each streamline for the appropriate choice of the constants.

( vi ) When  $\omega = B z \bar{z}$ ,  $B$  being real constant, then equation (11) is satisfied provided

$$\overline{A} = \frac{\iota \mu}{\rho} \ln \bar{z} + m_8 \quad (28)$$

where  $m_8$  is a complex constant.

According to equation (28), the stream function  $\psi$  becomes

$$\psi = -\frac{B}{16} (z \bar{z})^2 - \frac{\iota \mu}{\rho} \ln \left( \frac{z}{\bar{z}} \right) + t \quad (29)$$

where  $t = m_8 + \overline{m_8}$ .

The stream function  $\psi$  in equation (29) is represented graphically in Fig.7 with  $B = 1$ ,  $\mu = 12$ ,  $\rho = 1$ ,  $t = 2$  and  $x, y \in [-1, 10]$ .

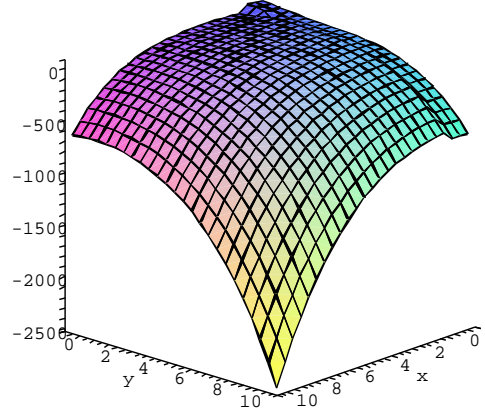


Figure No. 7: Graphical representation of equation (29).

## V. CONCLUSIONS

In this paper, we reconsidered the flow equations of Moro et al. [13] with the objective of determining some exact solutions.

For this purpose the vorticity function  $\omega$  and the stream function  $\psi$  are expressed in terms of complex variables and complex function. The condition which the complex functions must satisfy is determined through the equations for the generalized energy function  $h$  by using the compatibility condition  $\frac{\partial^2 h}{\partial x \partial y} = \frac{\partial^2 h}{\partial y \partial x}$ .

Some exact solutions to the flow equations are determined using the condition for the complex functions. The solutions presented in this paper admit, as particular cases, all the solutions of Moro et al. [13] by appropriately choosing the complex functions or the arbitrary constants therein.

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- [1] K. R. Rajagopal, *On the creeping flow of the second-order fluid*, J. Non-Newtonian Fluid Mech. **15**, 239–246 (1984).
  - [2] M. E. Erdogan, *Plane surface suddenly set in motion in a non-Newtonian fluid*, Acta Mech. **108**, 179–187 (1995).
  - [3] K. R. Rajagopal, A. S. Gupta, *An exact solution for a flow of a Steady flows of a non-Newtonian fluid past an infinite porous plate*, Meccanica **19**, 158–160 (1984).
  - [4] R. Bandelli, *Unsteady unidirectional flows of second grade fluids in domains with heated boundaries*, Int. J. Nonlinear Mech. **30**, 263–269 (1995).
  - [5] A. M. Siddiqui, P. N. Kaloni, *Certain inverse solutions of a non-Newtonian fluid*, Int. J. Nonlinear Mech. **21**, 459–473 (1986).
  - [6] A. M. Benharbit, A. M. Siddiqui, *Certain solutions of the equations of the planar motion of a second grade fluid for steady and unsteady cases*, Acta Mech. **94**, 85–96 (1992).
  - [7] P. D. Ariel, *The flow of a viscoelastic fluid past a porous plate*, Acta Mech. **107**, 199–204 (1994).
  - [8] P. D. Ariel, *Flow of a third grade fluid through a porous flat channel*, International Journal of Engineering Science **41**, 1267–1285 (2003).
  - [9] C. Fetecau, *Cone and plate flow of a second grade*, Acta Mech. **122**, 225–230 (1977).
  - [10] T. Hayat, A. H. Kara, and E. Momoniat, *Exact flow of a third grade fluid on a porous wall*, Int. J. Nonlinear Mech. **38**, 1533–1537 (2003).
  - [11] T. Hayat, Y. Wang, and K. Hutter, *Hall effects on the unsteady hydromagnetic oscillatory flow of a second grade fluid*, Int. J. Nonlinear Mech. **39**, 1027–1037 (2004).
  - [12] T. Hayat, M. Khan, A. M. Siddiqui and S. Asghar, *Transient flows of a second grade fluid*, Int. J. Nonlinear Mech. **39**, 1621–1633 (2004).
  - [13] L. Moro, A. M. Siddiqui and P. N. Kaloni, *Steady flows of a third grade fluid by transformation methods*, ZAMM **70**, No. 3, 189–198 (1990).
  - [14] R. S. Rivlin, J. L. Ericksen, *Stress-deformation relations for isotropic materials*, J. Rational Mech. Anal. **4**, 323–425 (1955).
  - [15] R. L. Fosdick, K. R. Rajagopal, *Thermodynamics and stability of fluids of third grade*, Proc. Roy. Soc. London A **339**, 351–377 (1980).
  - [16] C. Truesdell, W. Noll, *The non-linear field theories of mechanics*. In: *Handbuch der physik III*, No. 3, 494–513 (Springer, Berlin, 1965).
  - [17] M. P. Stallybrass, *A class of exact solution of the Navier-Stokes equations. Plane unsteady flow*, Lett. Appl. Engg. Sci. **21**, No. 2, 179–186 (1983).